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A New Inequality for Entire Functions

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We prove that, if $f(z)$ is an entire function and $|f(z)| \leqslant (A_1 + A_2 |z|^n) \exp[ax^2 + by^2 + cx + dy]$, then there are numbers $C_1, C_2 \geqslant 0$, depending only on n, A_1, A_2, a, b, c, and d such that $|f'(z)| \leqslant (C_1 + C_2 |z|^{n+1}) \exp(ax^2 + by^2 + cx + dy)$.

A classical theorem of S. Bernstein affirms that if $f(z)$ is an entire function of exponential type a and $|f(x)| \leqslant M$ for every real x, then $|f'(x)| \leqslant aM$ (cf. [1]). It is of interest to try to find similar inequalities for entire functions of order larger than one. Let $z = x + iy$; x, y real. The purpose of this paper is to prove the following:

THEOREM. *Let $f(z)$ be an entire function, $n \geqslant 0$ an integer, and A_1, A_2, a, b, c, and d real numbers with $A_1, A_2 \geqslant 0$ and $A_1 + A_2 > 0$. If $|f(z)| \leqslant (A_1 + A_2 |z|^n) \exp[ax^2 + by^2 + cx + dy]$ then there are numbers $C_1, C_2 \geqslant 0$, depending only on n, A_1, A_2, a, b, c, and d, such that $|f'(z)| \leqslant (C_1 + C_2 |z|^{n+1}) \exp(ax^2 + by^2 + cx + dy)$.*

COROLLARY. *Let $\|f(z)\|^* = \sup\{(A_1 + A_2 |z|^n)^{-1} \exp(-ax^2 - by^2 - cx - dy) \, |f(z)|\}$, where the supremum is taken over the set of all complex numbers. If $\|f(z)\|^* \leqslant 1$, then there are constants D_1 and D_2 depending only on n, A_1, A_2, a, b, c, and d, such that $\|(D_1 + D_2 |z|)^{-1} f'(z)\|^* \leqslant 1$.*

The idea of the proof is to first find a bound for the Fourier transform $\hat{f}(z)$ of $f(z)$. Setting $q(z) = z\hat{f}(z)$ we then find a bound for $\hat{q}(z)$. The proof is completed by noticing that $f'(z) = -i\hat{q}(-z)$. To make the proof of the theorem easier to follow we shall first prove two lemmas, of some independent interest.

LEMMA 1. *Let $f(z)$ be an entire function, $n \geqslant 0$ an integer, and A, a, b, c, and d real numbers with $A, a, b > 0$. If $|f(z)| < A |z|^n \exp(-ax^2 + by^2 + cx + dy)$, then $\hat{f}(z)$ is an entire function and*

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$$|\hat{f}(z)| \leqslant A\Gamma[(n+1)/2](2\pi)^{-1/2} M \left| \frac{ix}{2b} - \frac{y-c}{2a} + 1 \right|^n$$

$$\times \exp\left[-\frac{x^2}{4b} + \frac{(y-c)^2}{4a} + \frac{dx}{2b} \right],$$

where $M = a^{-1/2}$ *if* $a \geqslant 1$, *and* $M = a^{-(n+1)/2}$ *if* $a < 1$.

Proof. That $\hat{f}(z)$ is an entire function follows by an application of the theorems of Morera and Fubini as in, e.g., [3, p. 131]. We now use a refinement of an argument used by Gel'fand and Šilov in [2, p. 239] (see also [4]). By definition, $\hat{f}(z) = (2\pi)^{-1/2} \int_R \exp(izu) f(u)\, du$. Integrating along the rectangle with vertices at $(\pm\delta, 0)$ and $(\pm\delta, v)$, applying Cauchy's theorem, and making $\delta \to +\infty$, we conclude that

$$\hat{f}(z) = (2\pi)^{-1/2} \int_R \exp[iz(u+iv)] f(u+iv)\, du.$$

Thus

$$\begin{aligned} |\hat{f}(z)| &\leqslant A(2\pi)^{-1/2} \exp(-xv + bv^2 + dv) \\ &\quad \times \int_R |u+iv|^n \exp(-uy - au^2 + cu)\, du \\ &= A(2\pi)^{-1/2} \exp[-xv + bv^2 + dv + (y-c)^2/4a]\, q(z), \end{aligned} \qquad (1)$$

where

$$q(z) = \int_R |u+iv|^n \exp\{-a[u + (y-c)/2a]^2\}\, du$$

$$\leqslant \sum_{k=0}^{n} C(n,k)\, |iv - (y-c)/2a|^{n-k} I_k(y), \qquad \text{where}$$

$$\begin{aligned} I_k(y) &= \int_R |u + (y-c)/2a|^k \exp\{-a[u + (y-c)/2a]^2\}\, du \\ &= \int_R |s|^k \exp(-as^2)\, ds \\ &= \Gamma[(k+1)/2]\, a^{-(k+1)/2} \leqslant \Gamma[(n+1)/2]\, M. \end{aligned}$$

Thus, $q(z) \leqslant \Gamma[(n+1)/2]\, M \sum_{k=0}^{n} C(n,k)\, |iv - (y-c)/2a|^{n-k} = \Gamma[(n+1)/2]\, M\, |iv - (y-c)/2a + 1|^n$. Setting $v = x/2b$, the conclusion readily follows from (1). Q.E.D.

LEMMA 2. *Under the hypotheses of Lemma* 1 *there are constants* C_1 *and* C_2, *dependent only on* n, A_1, A_2, a, b, c, *and* d, *such that*

$$|f'(z)| < (C_1 + C_2 |z|^{n+1}) \exp[-ax^2 + by^2 + cx + dy].$$

Proof. From Lemma 1 we readily infer that there are constants B_1 and B_2, dependent only on n, A, a, b, c, and d, such that

$$|\hat{f}(z)| < (B_1 + B_2 |z|^n) \exp(-\alpha x^2 + \beta y^2 + \gamma x + \delta y),$$

where $\alpha = 1/4b$, $\beta = 1/4a$, $\gamma = d/2b$, and $\delta = -c/2a$.

Let $q(z) = z\hat{f}(z)$. Then

$$|q(z)| < (B_1 |z| + B_2 |z|^{n+1}) \exp(-\alpha x^2 + \beta y^2 + \gamma x + \delta y).$$

Applying Lemma 1 again, we readily conclude that there are constants C_1 and C_2, dependent only on n, A, a, b, c, and d, such that

$$|\hat{q}(z)| < (C_1 + C_2 |z|^{n+1}) \exp(-\alpha_1 x^2 + \beta_1 y^2 + \gamma_1 x + \delta_1 y),$$

where $\alpha_1 = 1/4\beta = a$, $\beta_1 = 1/4\alpha = b$, $\gamma_1 = \delta/2\beta = -c$, and $\delta_1 = -\gamma/2\alpha = -d$. Hence, $|\hat{q}(z)| \leqslant (C_1 + C_2 |z|^{n+1}) \exp(-ax^2 + by^2 - cx - dy)$. Since $f'(z) = -i\hat{q}(-z)$, the conclusion readily follows. Q.E.D.

Proof of Theorem. Let ε be an arbitrary positive real number (say $\varepsilon = |a| + |b| + 1$), and $g(z) = \exp[-(a+\varepsilon)z^2] f(z)$. Then $|g(z)| \leqslant (A_1 + A_2 |z|^n) \exp(-\varepsilon x^2 + (a+b+\varepsilon) y^2 + cx + dy)$, and therefore Lemma 2 readily implies that there are constants B_1 and B_2 such that

$$|g'(z)| \leqslant (B_1 + B_2 |z|^{n+1}) \times \exp(-\varepsilon x^2 + (a+b+\varepsilon) y^2 + cx + dy).$$

Since $f(z) = \exp[(a+\varepsilon)z^2] g(z)$, it is clear that $f'(z) = [2(a+\varepsilon) z g(z) + g'(z)] \exp[(a+\varepsilon) z^2]$. Thus, $|f'(z)| \leqslant [2(a+\varepsilon) |z| (A_1 + A_2 |z|^n) + (B_1 + B_2 |z|^{n+1})] \exp(ax^2 + by^2 + cx + dy)$, and the conclusion follows. Q.E.D.

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